11 Discrete-Time Fourier Transform

Recommended Problems

P11.1

Compute the discrete-time Fourier transform of the following signals.

(a) $x[n] = (\frac{1}{4})^n u[n]$

(b) $x[n] = (a^n \sin \Omega_0 n)u[n], \qquad |a| < 1$

(c) $x[n]$ as shown in Figure P11.1

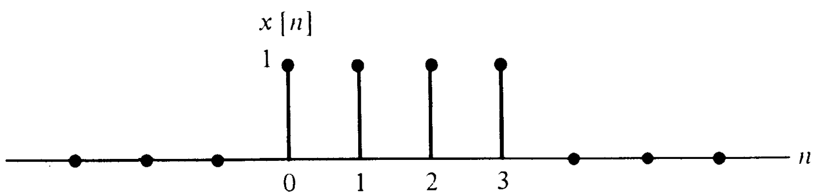

Figure P11.1

(d) $x[n] = (\frac{1}{4})^n u[n + 2]$

P11.2

(a) Consider the linear constant coefficient difference equation

$$y[n] - \tfrac{1}{2}y[n - 1] = x[n],$$

which describes a linear, time-invariant system initially at rest. What is the system function that describes $Y(\Omega)$ in terms of $X(\Omega)$?

(b) Using Fourier transforms, evaluate $y[n]$ if $x[n]$ is

(i) $\delta[n]$

(ii) $\delta[n - n_0]$

(iii) $(\frac{3}{4})^n u[n]$

P11.3

(a) Consider a system with impulse response

$$h[n] = \left[\left(\frac{1}{2}\right)^n \cos\frac{\pi n}{2}\right] u[n]$$

Determine the system transfer function $H(\Omega)$.

(b) Suppose that $x[n] = \cos(\pi n/2)$. Determine the system output $y[n]$ using the transfer function $H(\Omega)$ found in part (a).

P11.4

A particular LTI system is described by the difference equation

$$y[n] + \tfrac{1}{4}y[n - 1] - \tfrac{1}{8}y[n - 2] = x[n] - x[n - 1]$$

(a) Find the impulse response of the system.

(b) Evaluate the magnitude and phase of the system frequency response at $\Omega = 0$, $\Omega = \pi/4$, $\Omega = -\pi/4$, and $\Omega = 9\pi/4$.

P11.5

$x[n]$ is a finite-duration signal of length N so that $x[n] = 0$, $n < 0$ and $n > N - 1$. The discrete-time Fourier transform of $x[n]$ is denoted by $X(\Omega)$. We generate the periodic signal $\tilde{y}[n]$ by periodically replicating $x[n]$, i.e.,

$$\tilde{y}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$$

(a) Write the expression in terms of $x[n]$ for the Fourier series coefficients a_k of $\tilde{y}[n]$.

(b) Write an expression relating the Fourier series coefficients of $\tilde{y}[n]$ to the Fourier transform of $x[n]$.

P11.6

(a) Four different transforms have been introduced thus far:

I. Continuous-time Fourier series
II. Discrete-time Fourier series
III. Continuous-time Fourier transform
IV. Discrete-time Fourier transform

In the following table, fill in the blanks with I, II, III, or IV depending on which transform(s) can be used to represent the signal described on the left. Finite duration means that the signal is guaranteed to be nonzero over only a finite interval.

Signal Description			Transform
Continuous time	Infinite duration	Periodic	
Continuous time	Infinite duration	Aperiodic	
Continuous time	Finite duration	Aperiodic	
Discrete time	Infinite duration	Periodic	
Discrete time	Infinite duration	Aperiodic	
Discrete time	Finite duration	Aperiodic	

(b) Which of the transforms in the preceding table possess the duality property summarized in Sections 4.6.6 and 5.9.1 of the text?

(c) Which of the transforms are always periodic?

Optional Problems

P11.7

For continuous-time signals, we saw that

$$\text{if} \quad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega), \quad \text{then} \quad x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Is there a similar property for discrete-time signals? If so, what is it? If not, why not?

P11.8

If $x[n]$ and $X(\Omega)$ denote a sequence and its Fourier transform, determine in terms of $x[n]$ the sequence corresponding to

(a) $X(\Omega - \Omega_0)$

(b) $Re\{X(\Omega)\}$

(c) $Im\{X(\Omega)\}$

(d) $|X(\Omega)|^2$

Hint: Write your answer in terms of a convolution.

P11.9

Suppose we have an LTI system characterized by an impulse response

$$h[n] = \frac{\sin \frac{\pi n}{3}}{\pi n}$$

(a) Sketch the magnitude of the system transfer function.

(b) Evaluate $y[n] = x[n] * h[n]$ when

$$x[n] = (-1)^n \cos \frac{3\pi}{4} n$$

P11.10

A particular discrete-time system has input $x[n]$ and output $y[n]$. The Fourier transforms of these signals are related by the following equation:

$$Y(\Omega) = 2X(\Omega) + e^{-j\Omega}X(\Omega) - \frac{dX(\Omega)}{d\Omega}$$

(a) Is the system linear? Clearly justify your answer.

(b) Is the system time-invariant? Clearly justify your answer.

(c) What is $y[n]$ if $x[n] = \delta[n]$?

P11.11

Consider a discrete-time sequence $\tilde{x}[n]$ that is periodic with period N. We know that $\tilde{x}[n]$ can be written as

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

(a) Show that by multiplying both sides of the equation by $e^{-jl(2\pi/N)n}$ and summing over one period, the discrete-time Fourier series coefficients a_k are obtained as

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

(b) The synthesis equation for an aperiodic discrete-time signal can be written as

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n}\, d\Omega$$

(i) Show that by multiplying both sides by $e^{-j\Omega_1 n}$ and summing over $n = -\infty$ to $n = \infty$,

$$\sum_{n=-\infty}^{\infty} x[n] e^{j\Omega_1 n} = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \sum_{n=-\infty}^{\infty} e^{j(\Omega - \Omega_1)n}\, d\Omega$$

(ii) Show that

$$\sum_{n=-\infty}^{\infty} e^{j(\Omega - \Omega_1)n} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - \Omega_1 + 2\pi n)$$

Hint: Consider $\sum_{n=-\infty}^{\infty} e^{j(\Omega - \Omega_1)n}$ as the Fourier series representation of some continuous-time periodic function.

(iii) By combining the results of parts (i) and (ii), establish that

$$\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = X(\Omega)$$

P11.12

The Fourier transform of a discrete-time periodic signal is based on the fact that such a series can be written as

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

(a) Establish that the Fourier transform of $e^{jk(2\pi/N)n}$ is

$$\sum_{n=-\infty}^{\infty} 2\pi\delta\left(\Omega - \frac{2\pi k}{N} + 2\pi n\right)$$

(b) Establish that the Fourier transform of

$$\sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

is

$$\sum_{n=-\infty}^{\infty} 2\pi \sum_{k=\langle N \rangle} a_k \delta\left(\Omega - \frac{2\pi k}{N} + 2\pi n\right)$$

(c) Establish that

$$\sum_{n=-\infty}^{\infty} 2\pi \sum_{k=\langle N\rangle} a_k \delta\left(\Omega - \frac{2\pi k}{N} + 2\pi n\right) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right),$$

which shows that

$$\tilde{x}[n] \xleftrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right)$$

(d) Use the result in part (c) to verify that the Fourier series coefficients

$$a_k = \frac{1}{N} X(\Omega) \Big|_{\Omega=(2\pi k)/N}$$

where $X(\Omega)$ is the Fourier transform of $x[n]$, which consists of a single period of $\tilde{x}[n]$.