5 Properties of Linear, Time-Invariant Systems

Recommended Problems

P5.1

Consider an integrator that has the input-output relation

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Determine the input-output relation for the inverse system.

P5.2

The first-order difference equation $y[n] - ay[n-1] = x[n]$, $0 < a < 1$, describes a particular discrete-time system initially at rest.

(a) Verify that the impulse response $h[n]$ for this system is $h[n] = a^n u[n]$.

(b) Is the system

(i) memoryless?

(ii) causal?

(iii) stable?

Clearly state your reasoning.

(c) Is this system stable if $|a| > 1$?

P5.3

The first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

describes a particular continuous-time system initially at rest.

(a) Verify that the impulse response of this system is $h(t) = e^{-2t}u(t)$.

(b) Is this system

(i) memoryless?

(ii) causal?

(iii) stable?

Clearly state your reasoning.

P5.4

Consider the linear, time-invariant system in Figure P5.4, which is composed of a cascade of two LTI systems. $u(t)$ is a unit step signal and $s(t)$ is the step response of system L.

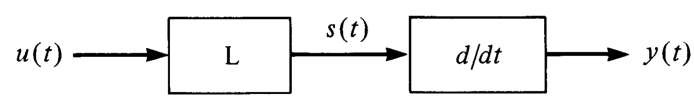

Figure P5.4

Using the fact that the overall response of LTI systems in cascade is independent of the order in which they are cascaded, show that the impulse response of system L is the derivative of its step response, i.e.,

$$h(t) = \frac{ds(t)}{dt}$$

P5.5

Consider the cascade of two systems shown in Figure P5.5. System B is the inverse of system A.

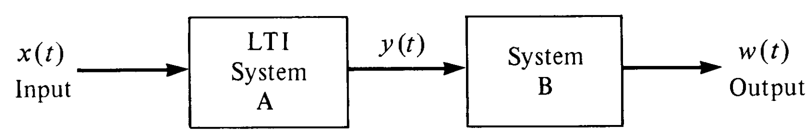

Figure P5.5

(a) Suppose the input is $\delta(t)$. What is the output $w(t)$?

(b) Suppose the input is some more general signal $x(t)$. What is the output $w(t)$ in terms of $x(t)$?

Optional Problems

P5.6

(a) Consider again the cascade of two systems presented in Problem P5.5. Suppose an input $x_1(t)$ produces $y_1(t)$ as system A output and an input $x_2(t)$ produces $y_2(t)$ as system A output. What is $w(t)$ if the input is such that $y(t)$, the output of system A, is $ay_1(t) + by_2(t)$ with a, b constants?

(b) Suppose an input $x_1(t)$ produces $y_1(t)$ as system A output. What is $w(t)$ if $x(t)$ is such that $y(t) = y_1(t - \tau)$?

(c) Is system B an LTI system? Justify your answer.

P5.7

Consider the three discrete-time signals shown in Figure P5.7.

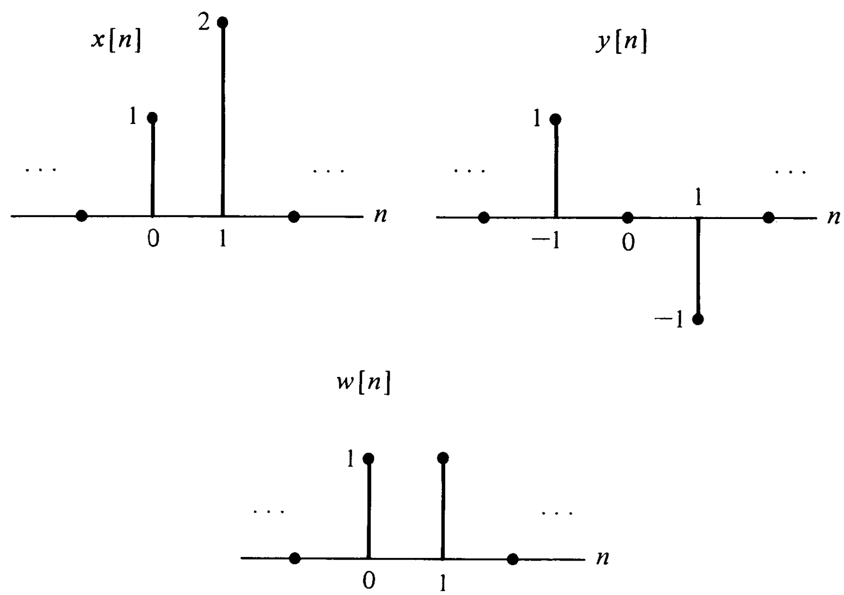

Figure P5.7

(a) Verify the distributive law of convolution:

$$(x + w) * y = (x * y) + (w * y)$$

(b) You may have noticed a similarity between the convolution operation and multiplication, but they are *not* equivalent. Verify that

$$(x * y) \cdot w \neq x * (y \cdot w)$$

P5.8

Let $y(t) = x(t) * h(t)$. Show the following.

(a) $\dfrac{dy(t)}{dt} = x(t) * \dfrac{dh(t)}{dt} = \dfrac{dx(t)}{dt} * h(t)$

(b) $y(t) = (\int_{-\infty}^{t} x(\tau)\, d\tau) * h'(t)$

(c) $y(t) = \int_{-\infty}^{t} [x'(\tau) * h(\tau)]\, d\tau$

(d) $y(t) = x'(t) * \int_{-\infty}^{t} h(\tau)\, d\tau$

P5.9

Determine if each of the following statements concerning LTI systems is true or false. Justify your answers.

(a) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is periodic and nonzero, the system is unstable.

(b) The inverse of a causal LTI system is always causal.

(c) If $|h[n]| \leq K$ for each n, where K is a given number, then the LTI system with $h[n]$ as its impulse response is stable.

(d) If a discrete-time LTI system has an impulse response $h[n]$ of finite duration, the system is stable.

(e) If an LTI system is causal, it is stable.

(f) The cascade of a noncausal LTI system with a causal one is necessarily noncausal.

(g) A continuous-time LTI system is stable if and only if its step response $s(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{+\infty} |s(t)|\, dt < \infty$$

(h) A discrete-time LTI system is causal if and only if its step response $s[n]$ is zero for $n < 0$.

P5.10

In Section 3.7 of the text we characterized the unit doublet through the equation

$$x(t) * u_1(t) = \int_{-\infty}^{+\infty} x(t - \tau)u_1(\tau)\, d\tau = x'(t) \qquad \text{(P5.10-1)}$$

for any signal $x(t)$. From this equation we derived the fact that

$$\int_{-\infty}^{+\infty} g(\tau)u_1(\tau)\, d\tau = -g'(0) \qquad \text{(P5.10-2)}$$

(a) Show that eq. (P5.10-2) is an equivalent characterization of $u_1(t)$ by showing that eq. (P5.10-2) implies eq. (P5.10-1). [*Hint:* Fix t and define the signal $g(\tau) = x(t - \tau)$.]

Thus we have seen that characterizing the unit impulse or unit doublet by how it behaves under convolution is equivalent to characterizing how it behaves under integration when multiplied by an arbitrary signal $g(t)$. In fact, as indicated in Section 3.7 of the text, the equivalence of these operational definitions holds for all signals and in particular for all singularity functions.

(b) Let $f(t)$ be a given signal. Show that

$$f(t)u_1(t) = f(0)u_1(t) - f'(0)\delta(t)$$

by showing that both have the same operational definitions.

(c) Determine the value of

$$\int_{-\infty}^{\infty} x(\tau)u_2(\tau)\, d\tau$$

(d) Find an expression for $f(t)u_2(t)$ analogous to that considered in part (b) for $f(t)u_1(t)$.

P5.11

Consider the cascade of two systems H and G as shown in Figure P5.11.

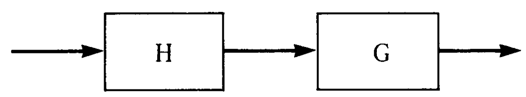

Figure P5.11

(a) If H and G are both LTI causal systems, prove that the overall system is causal.

(b) If H and G are both stable systems, show that the overall system is stable.

P5.12

Find the combined impulse response of the LTI system in Figure P5.12. Recall that $x(t) * h(t) * h^{-1}(t) = x(t)$.

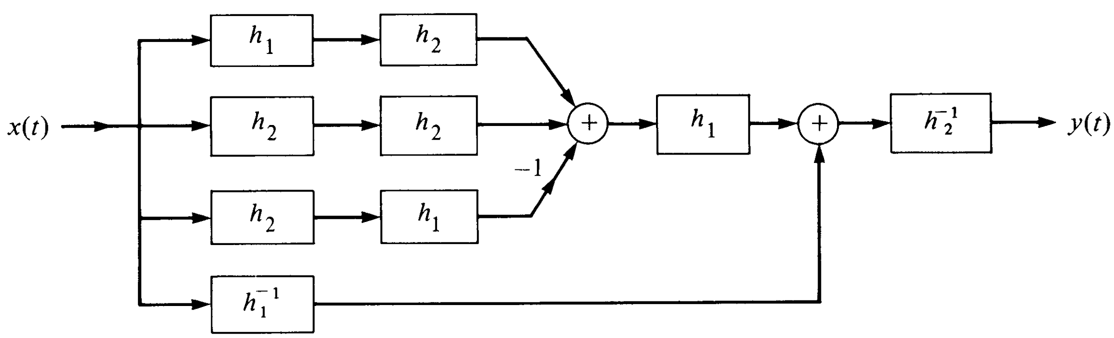

Figure P5.12

P5.13

Find the necessary and sufficient condition on the impulse response $h[n]$ such that for *any* input $x[n]$,

$$\max\{|x[n]|\} \geq \max\{|y[n]|\},$$

where $y[n] = x[n] * h[n]$.