6 Systems Represented by Differential and Difference Equations

Recommended Problems

P6.1

Suppose that $y_1(t)$ and $y_2(t)$ *both* satisfy the homogeneous linear constant-coefficient differential equation (LCCDE)

$$\frac{dy(t)}{dt} + ay(t) = 0$$

Show that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$, where α and β are any two constants, is also a solution to the homogeneous LCCDE.

P6.2

In this problem, we consider the homogeneous LCCDE

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0 \qquad \text{(P6.2-1)}$$

(a) Assume that a solution to eq. (P6.2-1) is of the form $y(t) = e^{st}$. Find the quadratic equation that s must satisfy, and solve for the possible values of s.

(b) Find an expression for the family of signals $y(t)$ that will satisfy eq. (P6.2-1).

P6.3

Consider the LCCDE

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = x(t), \qquad x(t) = e^{-t}u(t) \qquad \text{(P6.3-1)}$$

(a) Determine the family of signals $y(t)$ that satisfies the associated homogeneous equation.

(b) Assume that for $t > 0$, one solution of eq. (P6.3-1), with $x(t)$ as specified, is of the form

$$y_1(t) = Ae^{-t}, \qquad t > 0$$

Determine the value of A.

(c) By substituting into eq. (P6.3-1), show that

$$y_1(t) = [2e^{-t/2} - 2e^{-t}]u(t)$$

is one solution for all t.

P6.4

Consider the block diagram relating the two signals $x[n]$ and $y[n]$ given in Figure P6.4.

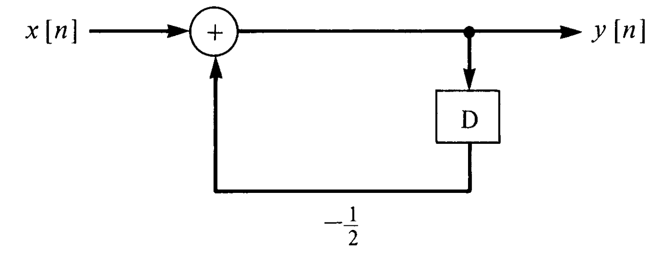

Figure P6.4

Assume that the system described in Figure P6.4 is causal and is initially at rest.

(a) Determine the difference equation relating $y[n]$ and $x[n]$.

(b) Without doing any calculations, determine the value of $y[-5]$ when $x[n] = u[n]$.

(c) Assume that a solution to the difference equation in part (a) is given by

$$y[n] = K\alpha^n u[n]$$

when $x[n] = \delta[n]$. Find the appropriate value of K and α, and verify that $y[n]$ satisfies the difference equation.

(d) Verify your answer to part (c) by directly calculating $y[0]$, $y[1]$, and $y[2]$.

P6.5

Figure P6.5 presents the direct form II realization of a difference equation. Assume that the resulting system is linear and time-invariant.

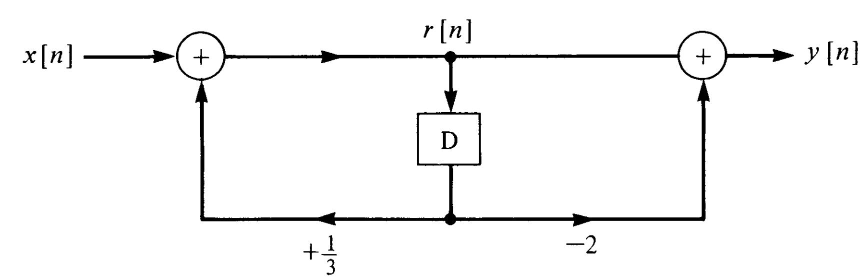

Figure P6.5

(a) Find the direct form I realization of the difference equation.

(b) Find the difference equation described by the direct form I realization.

(c) Consider the intermediate signal $r[n]$ in Figure P6.5.

(i) Find the relation between $r[n]$ and $y[n]$.

(ii) Find the relation between $r[n]$ and $x[n]$.

(iii) Using your answers to parts (i) and (ii), verify that the relation between $y[n]$ and $x[n]$ in the direct form II realization is the same as your answer to part (b).

P6.6

Consider the following differential equation governing an LTI system.

$$\frac{dy(t)}{dt} + ay(t) = b\,\frac{dx(t)}{dt} + cx(t) \qquad \text{(P6.6-1)}$$

(a) Draw the direct form I realization of eq. (P6.6-1).

(b) Draw the direct form II realization of eq. (P6.6-1).

Optional Problems

P6.7

Consider the block diagram in Figure P6.7. The system is causal and is initially at rest.

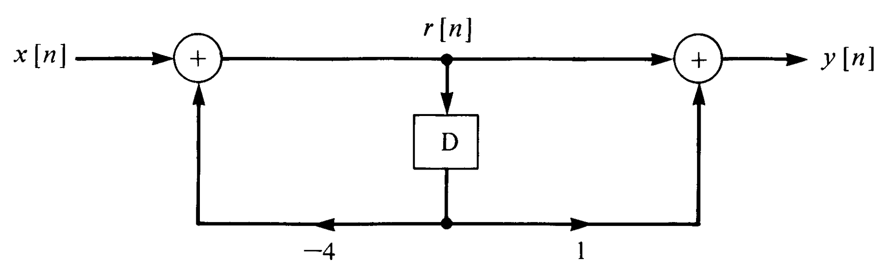

Figure P6.7

(a) Find the difference equation relating $x[n]$ and $y[n]$.

(b) For $x[n] = \delta[n]$, find $r[n]$ for all n.

(c) Find the system impulse response.

P6.8

Consider the system shown in Figure P6.8. Find the differential equation relating $x(t)$ and $y(t)$.

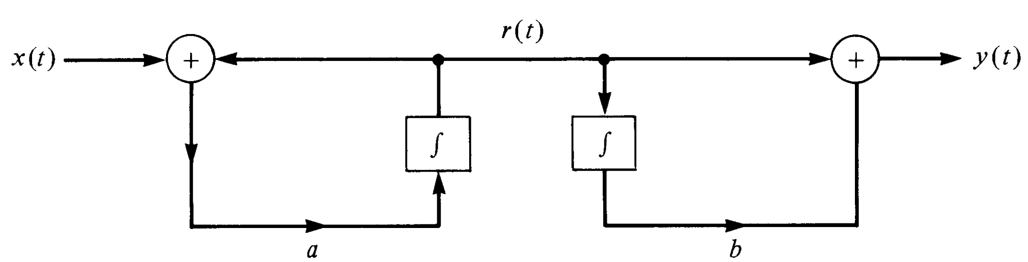

Figure P6.8

P6.9

Consider the following difference equation:

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] \tag{P6.9-1}$$

with

$$x[n] = K(\cos \Omega_0 n)u[n] \tag{P6.9-2}$$

Assume that the solution $y[n]$ consists of the sum of a particular solution $y_p[n]$ to eq. (P6.9-1) for $n \geq 0$ and a homogeneous solution $y_h[n]$ satisfying the equation $y_h[n] - \frac{1}{2}y_h[n-1] = 0$.

(a) If we assume that $y_h[n] = Az_0^n$, what value must be chosen for z_0?

(b) If we assume that for $n \geq 0$,

$$y_p[n] = B\cos(\Omega_0 n + \theta),$$

what are the values of B and θ? [*Hint:* It is convenient to view $x[n] = Re\{Ke^{j\Omega_0 n}u[n]\}$ and $y[n] = Re\{Ye^{j\Omega_0 n}u[n]\}$, where Y is a complex number to be determined.]

P6.10

Show that if $r(t)$ satisfies the homogeneous differential equation

$$\sum_{i=1}^{M} \frac{d^i r(t)}{dt^i} = 0$$

and if $s(t)$ is the response of an arbitrary LTI system H to the input $r(t)$, then $s(t)$ satisfies the same homogeneous differential equation.

P6.11

(a) Consider the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0 \tag{P6.11-1}$$

Show that if s_0 is a solution of the equation

$$p(s) = \sum_{k=0}^{N} a_k s^k = 0, \tag{P6.11-2}$$

then $Ae^{s_0 t}$ is a solution of eq. (P6.11-1), where A is an arbitrary complex constant.

(b) The polynomial $p(s)$ in eq. (P6.11-2) can be factored in terms of its roots $s_1, \ldots, s_r$:

$$p(s) = a_N(s - s_1)^{\sigma_1}(s - s_2)^{\sigma_2} \cdots (s - s_r)^{\sigma_r},$$

where the s_i are the distinct solutions of eq. (P6.11-2) and the σ_i are their *multiplicities.* Note that

$$\sigma_1 + \sigma_2 + \cdots + \sigma_r = N$$

In general, if $\sigma_i > 1$, then not only is $Ae^{s_i t}$ a solution of eq. (P6.11-1) but so is $At^j e^{s_i t}$ as long as j is an integer greater than or equal to zero and less than or equal to $\sigma_i - 1$. To illustrate this, show that if $\sigma_i = 2$, then $Ate^{s_i t}$ is a solution of eq. (P6.11-1). [*Hint:* Show that if s is an arbitrary complex number, then

$$\sum_{k=0}^{N} a_k \frac{d^k(Ate^{st})}{dt^k} = Ap(s)te^{st} + A\frac{dp(s)}{ds}e^{st}]$$

Thus, the most general solution of eq. (P6.11-1) is

$$\sum_{i=1}^{p} \sum_{j=0}^{\sigma_i - 1} A_{ij} t^j e^{s_i t},$$

where the A_{ij} are arbitrary complex constants.

(c) Solve the following homogeneous differential equation with the specified auxiliary conditions.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 0, \qquad y(0) = 1, \quad y'(0) = 1$$