8 Continuous-Time Fourier Transform

Recommended Problems

P8.1

Consider the signal $x(t)$, which consists of a single rectangular pulse of unit height, is symmetric about the origin, and has a total width T_1.

(a) Sketch $x(t)$.

(b) Sketch $\tilde{x}(t)$, which is a periodic repetition of $x(t)$ with period $T_0 = 3T_1/2$.

(c) Compute $X(\omega)$, the Fourier transform of $x(t)$. Sketch $|X(\omega)|$ for $|\omega| \leq 6\pi/T_1$.

(d) Compute a_k, the Fourier series coefficients of $\tilde{x}(t)$. Sketch a_k for $k = 0, \pm 1, \pm 2, \pm 3$.

(e) Using your answers to (c) and (d), verify that, for this example,

$$a_k = \frac{1}{T_0} X(\omega) \Big|_{\omega=(2\pi k)/T_0}$$

(f) Write a statement that indicates how the Fourier series for a periodic function can be obtained if the Fourier transform of one period of this periodic function is given.

P8.2

Find the Fourier transform of each of the following signals and sketch the magnitude and phase as a function of frequency, including both positive and negative frequencies.

(a) $\delta(t - 5)$

(b) $e^{-at}u(t)$, $\quad a$ real, positive

(c) $e^{(-1+j2)t}u(t)$

P8.3

In this problem we explore the definition of the Fourier transform of a periodic signal.

(a) Show that if $x_3(t) = ax_1(t) + bx_2(t)$, then $X_3(\omega) = aX_1(\omega) + bX_2(\omega)$.

(b) Verify that

$$e^{j\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\, d\omega$$

From this observation, argue that the Fourier transform of $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$.

(c) Recall the synthesis equation for the Fourier series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

By taking the Fourier transform of both sides and using the results to parts (a) and (b), show that

$$\tilde{X}(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

(d) Sketch $\tilde{X}(\omega)$ for your answer to Problem P8.1(d) for $|\omega| \leq 4\pi/T_0$.

P8.4

(a) Consider the often-used alternative definition of the Fourier transform, which we will call $X_a(f)$. The forward transform is written as

$$X_a(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt,$$

where f is the frequency variable in hertz. Derive the inverse transform formula for this definition. Sketch $X_a(f)$ for the signal discussed in Problem P8.1.

(b) A second, alternative definition is

$$X_b(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-jvt}\,dt$$

Find the inverse transform relation.

P8.5

Consider the periodic signal $\tilde{x}(t)$ in Figure P8.5-1, which is composed solely of impulses.

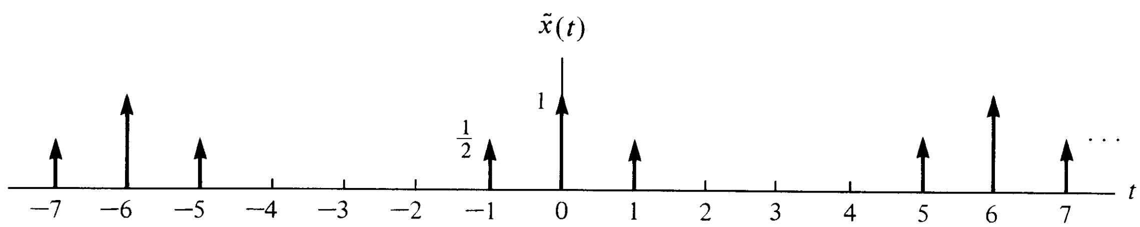

Figure P8.5-1

(a) What is the fundamental period T_0?

(b) Find the Fourier series of $\tilde{x}(t)$.

(c) Find the Fourier transform of the signals in Figures P8.5-2 and P8.5-3.

(i)

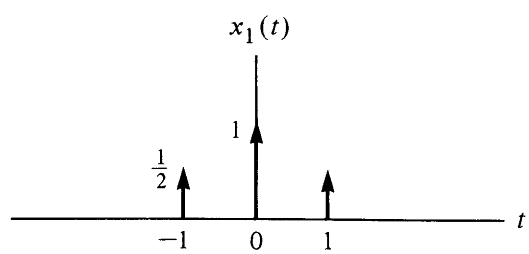

Figure P8.5-2

(ii)

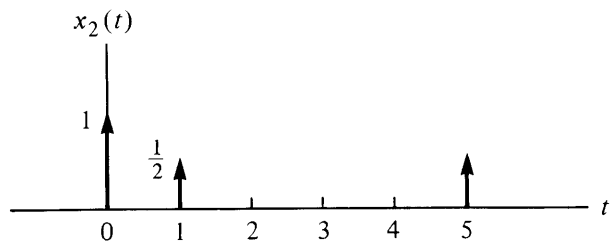

Figure P8.5-3

(d) $\tilde{x}(t)$ can be expressed as either $x_1(t)$ periodically repeated or $x_2(t)$ periodically repeated, i.e.,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_1(t - kT_1), \qquad \text{or} \tag{P8.5-1}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_2(t - kT_2) \tag{P8.5-2}$$

Determine T_1 and T_2 and demonstrate graphically that eqs. (P8.5-1) and (P8.5-2) are valid.

(e) Verify that the Fourier series of $\tilde{x}(t)$ is composed of scaled samples of either $X_1(\omega)$ or $X_2(\omega)$.

P8.6

Find the signal corresponding to the following Fourier transforms.

(a) $X_a(\omega) = \dfrac{1}{7 + j\omega}$

(b)

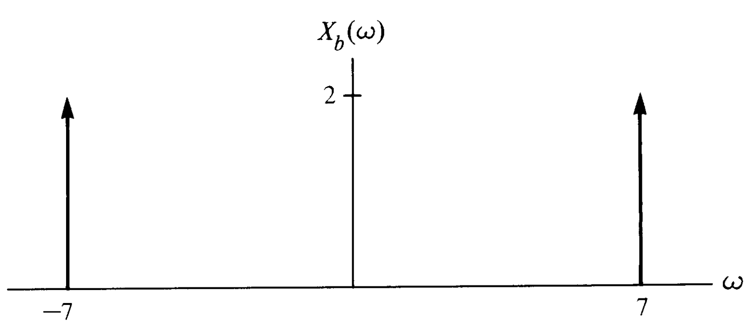

Figure P8.6-1

(c) $X_c(\omega) = \dfrac{1}{9 + \omega^2}$

See Example 4.8 in the text (page 191).

(d) $X_d(\omega) = X_a(\omega)X_b(\omega)$, where $X_a(\omega)$ and $X_b(\omega)$ are given in parts (a) and (b), respectively. Try to simplify as much as possible.

(e)

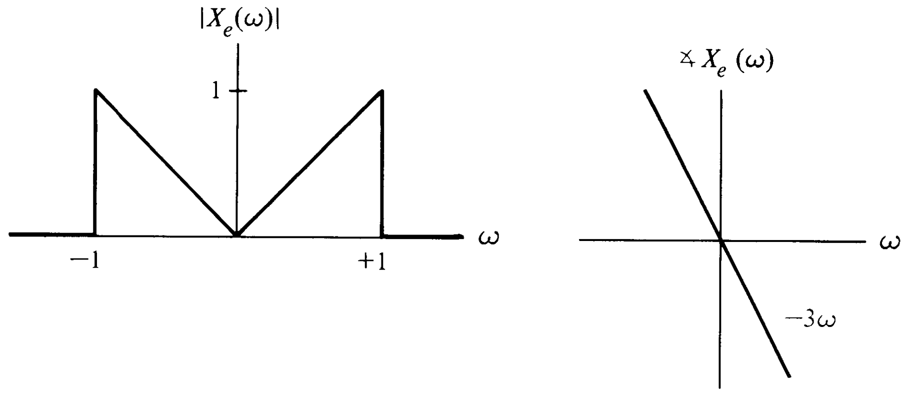

Figure P8.6-2

Optional Problems

P8.7

In earlier lectures, the response of an LTI system to an input $x(t)$ was shown to be $y(t) = x(t) * h(t)$, where $h(t)$ is the system impulse response.

(a) Using the fact that

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau,$$

show that

$$Y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)e^{-j\omega t}\, d\tau\, dt$$

(b) By appropriate change of variables, show that

$$Y(\omega) = X(\omega)H(\omega),$$

where $X(\omega)$ is the Fourier transform of $x(t)$, and $H(\omega)$ is the Fourier transform of $h(t)$.

P8.8

Consider the impulse train

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

shown in Figure P8.8-1.

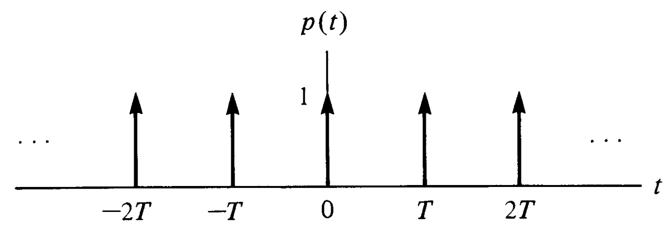

Figure P8.8-1

(a) Find the Fourier series of $p(t)$.

(b) Find the Fourier transform of $p(t)$.

(c) Consider the signal $x(t)$ shown in Figure P8.8-2, where $T_1 < T$.

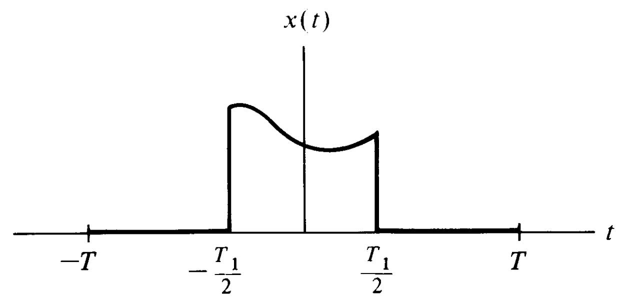

Figure P8.8-2

Show that the periodic signal $\tilde{x}(t)$, formed by periodically repeating $x(t)$, satisfies

$$\tilde{x}(t) = x(t) * p(t)$$

(d) Using the result to Problem P8.7 and parts (b) and (c) of this problem, find the Fourier transform of $\tilde{x}(t)$ in terms of the Fourier transform of $x(t)$.